Convergence of martingales via enriched dagger categories

Paolo Perrone and Ruben Van Belle

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1. Martingales
2. The category **Krn** and functors L^n
    1. Dagger structure
    2. Idempotents
    3. Martingale convergence I
    4. Topological enrichment
    5. Idempotent Levi property
    6. Martginale convergence II

Conditional expectation

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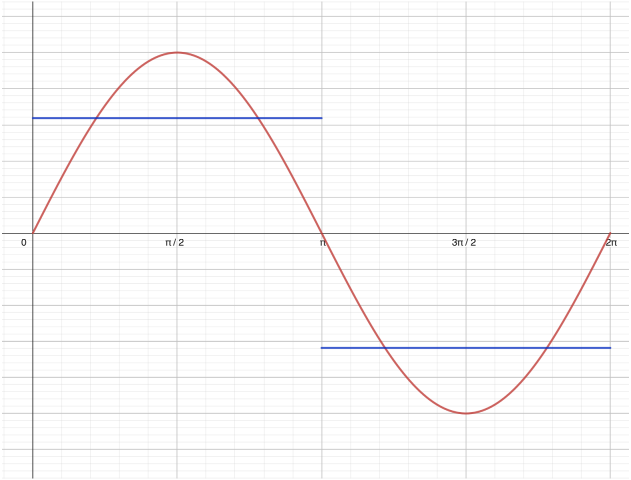
The **conditional expectation of** f is a $\mathcal{B}$-measurable integrable map $\mathbb{E}[f \mid \mathcal{B}] : X \to \mathbb{R}$ such that for all $B \in \mathcal{B}$,

$$\int_B f(x)p(\mathrm{d}x) = \int_B \mathbb{E}[f \mid \mathcal{B}](x)p(\mathrm{d}x).$$

Conditional expectation

Example: $f : [0, 2\pi] \to \mathbb{R} : x \mapsto \sin(x)$ and $\mathcal{B} = \{\emptyset, [0, \pi), [\pi, 2\pi], [0, 2\pi]\}$.



Martingales

Let I be a directed poset. Let $(X, \mathcal{A}, p)$ be a probability space and let $(\mathcal{B}_i)_{i \in I}$ be an increasing net of σ-subalgebras of $\mathcal{A}$. (*filtered probability space*).

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A **martingale** is a collection of integrable maps $(f_i : X \to \mathbb{R})$ such that f_i is $\mathcal{B}_i$-measurable and such that

$$f_i = \mathbb{E}[f_j \mid \mathcal{B}_i] \quad \text{for } i \leq j.$$

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Remark: If the collection of σ-subalgebras is decreasing, we talk about a *backwards filtered probability space* and *backwards martingales*.

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Let $n \in [2, \infty)$.

Theorem

Let $(f_i)_i$ be an L^n-bounded martingale on a filtered probability space $(X, (\mathcal{B}_i)_i, \mathcal{A}, p)$, then there exists an $f \in L^n(X, \mathcal{A}, p)$ such that:

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Remark: The result says something about both a categorical limit (common refinement) and a topological limit.

The category **Krn** and the functors L^n

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Two Markov kernels $k_1, k_2 : X \to Y$ are p-**almost surely equal** if for every $B \in \mathcal{B}$

$$k_1(B \mid -) = k_2(B \mid -) \quad p\text{-almost surely.}$$

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The category $\mathbf{Hilb}_{\leq 1}$ is the full subcategory of Hilbert spaces .

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$$L^n : \mathbf{Krn}^{\mathrm{op}} \to \mathbf{Ban}_{\leq 1}$$

on objects as follows:

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For a measure-preserving Markov kernel $k : (X, \mathcal{A}, p) \to (Y, \mathcal{B}, q)$ and $f \in L^n(Y, \mathcal{B}, q)$, define

$$k^* f : X \to \mathbb{R} : x \mapsto \int_X f(y) k(\mathrm{d}y \mid x).$$

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Remark: In the case $n = 2$, this functor factors through $\mathbf{Hilb}_{\leq 1}$.

The dagger structure of **Krn**

Definition

A **dagger category** is a category $\mathcal{C}$ together with a functor $(-)^+ : \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$ which is the identity on objects and such that $(f^+)^+ = f$.

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Example: $\mathbf{Hilb}_{\leq 1}$ becomes a dagger category via adjoints of 1-Lipschitz linear maps.

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$$\int_A k(B \mid x) p(\mathrm{d}x) = \int_B k^+(A \mid y) q(\mathrm{d}y)$$

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is called a **Bayesian inverse of** k.

If $(X, \mathcal{A}, p)$ and $(Y, \mathcal{B}, q)$ are (essentially) standard Borel probability spaces, then every measure-preserving kernel has an almost surely unique Bayesian inverse. (Rohklin's disinitegration theorem)

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The dagger structure of L^2

The functor $L^2 : \mathbf{Krn}^{\mathrm{op}} \to \mathbf{Hilb}_{\leq 1}$ preserves the dagger structure, i.e.

$$\begin{tikzcd} \mathbf{Krn}^{\mathrm{op}} \arrow[r, "L^2"] \arrow[d, "(-)^+"'] & \mathbf{Hilb}_{\leq 1} \arrow[d, "(-)^+"] \\ \mathbf{Krn} \arrow[r, "L^2"'] & \mathbf{Hilb}_{\leq 1}^{\mathrm{op}} \end{tikzcd}$$

For a measure-preserving kernel $k : (X, \mathcal{A}, p) \to (Y, \mathcal{B}, q)$, we have that for every $f \in L^2(X, \mathcal{A}, p)$ and $g \in L^2(Y, \mathcal{B}, q)$

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In particular, for $\pi : (X, \mathcal{A}, p) \to (X, \mathcal{B}, p)$ and $g = 1_B$ for some $B \in \mathcal{B}$, we find

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Therefore $(\pi^+)^* f = \mathbb{E}[f \mid \mathcal{B}]$.

Overview

	Dagger	Idempotents	Order	$\bigvee$	$\bigwedge$
Krn	Bayesian inverse				
$\mathbf{Hilb}_{\leq 1}$	adjoints				
L^2	✓				

	Enrichment	Levi property
Krn		
$\mathbf{Hilb}_{\leq 1}$		
L^2		
$\mathbf{Ban}_{\leq 1}$		
L^n		

Idempotents in $\mathbf{Hilb}_{\leq 1}$ and $\mathbf{Krn}$

Definition

An endomorphism $e : X \to X$ in a dagger category $\mathcal{C}$ is a **dagger idempotent** if

$$e \circ e = e = e^{+}.$$

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A **dagger splitting** of a dagger idempotent in a dagger category $\mathcal{C}$ is a pair of morphisms $(\pi : X \to A, \iota : A \to X)$ such that

$$\pi \circ \iota = \mathrm{id}_A \quad , \quad \iota \circ \pi = e \quad \text{and} \quad \pi^{+} = \iota.$$

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Example: Let X be a Hilbert space and let $e : X \to X$ be a dagger idempotent.

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Example: Let X be a Hilbert space and let $e : X \to X$ be a dagger idempotent. Then e has to be the orthogonal projection $p_{\mathrm{Im}(e)}$.

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$$\{\text{Dagger Idempotents}\} \cong \{\text{Closed subspaces}\}$$

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Example: Let $(X, \mathcal{A}, p)$ be an essentially standard Borel probability space and let $\mathcal{B} \subseteq \mathcal{A}$ be a σ-subalgebra. Let $\pi : (X, \mathcal{A}, p) \to (X, \mathcal{B}, p)$ be the setwise identity, then $e_{\mathcal{B}} := \pi^+ \circ \pi$ is a dagger idempotent.

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$$e_{(-)} : \{\sigma\text{-subalgebras}\} \to \{\text{Dagger idempotents}\}\,.$$

Proposition

The map $e_{(-)}$ is surjective.

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$$\{\text{eq. classes of } \sigma\text{-subalgebras}\} \cong \{\text{Dagger idempotents}\}$$

Where $\mathcal{B}_1 \sim \mathcal{B}_2$ if and only if $e_{\mathcal{B}_1} = e_{\mathcal{B}_2}$.

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Where $\mathcal{B}_1 \sim \mathcal{B}_2$ if and only if $e_{\mathcal{B}_1} = e_{\mathcal{B}_2}$. Every equivalence class $[\mathcal{B}]$ has a finest element, the *invariant σ-algebra* $\mathcal{I}_{e_{\mathcal{B}}}$ [2].

Ordering of idempotents in $\mathbf{Hilb}_{\leq 1}$ and $\mathbf{Krn}$

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For dagger idempotents $e_1, e_2 : X \to X$ in a dagger category $\mathcal{C}$, we write $e_1 \sqsubseteq e_2$ if and only if $e_1 \circ e_2 = e_1$.

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Example: For closed subspaces A_1 and A_2 of a Hilbert space X, we see that

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if and only if $A_1 \subseteq A_2$. We have an isomorphism of posets:

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Let $(X, \mathcal{A}, p)$ be an essentially standard Borel probability space and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be σ-subalgebras.
We say that $\mathcal{B}_1$ is **almost surely courser** than $\mathcal{B}_2$ if for every $B_1 \in \mathcal{B}_1$, there exists a $B_2 \in \mathcal{B}_2$ such that

$$p(B_1 \triangle B_2) = 0.$$

We write $\mathcal{B}_1 \lesssim \mathcal{B}_2$.

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Remark: $\mathcal{B}_1 \subseteq \mathcal{B}_2 \Rightarrow \mathcal{B}_1 \lesssim \mathcal{B}_2$

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Proposition

We have that $e_{\mathcal{B}_1} \circ e_{\mathcal{B}_2} = e_{\mathcal{B}_1}$ if and only if $\mathcal{B}_1 \lesssim \mathcal{B}_2$ if and only if $\mathcal{I}_{e_{\mathcal{B}_1}} \subseteq \mathcal{I}_{e_{\mathcal{B}_2}}$.

This induces an isomorphism of posets:

$$(\{\text{eq. classes of } \sigma\text{-subalgebras}\}, \lesssim) \cong (\{\text{Dagger idempotents}\}, \sqsubseteq).$$

Directed suprema and infima of idempotents

Example:

- Let X be a Hilbert space and consider a cofiltered collection $(e_i : X \to X)_i$ of dagger idempotents.

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- Similarly, for a filtered collection $(e_i)_i$, we have $\bigwedge_i e_i = p_{\bigcap_i A_i}$.

Directed suprema and infima of idempotents

Let $(X, \mathcal{A}, p)$ be an essentially standard Borel probability space.

Proposition

- Let $(\mathcal{B}_i)_i$ be a cofiltered collection of σ-subalgebras of $\mathcal{A}$ ordered by inclusion,then
$$\bigvee_i e_{\mathcal{B}_i} = e_{\sigma\left(\bigcup_i \mathcal{B}_i\right)}.$$
- Let $(\mathcal{B}_n)_n$ be a decreasing sequence of σ-subalgebras of $\mathcal{A}$ ordered by inclusion , then
$$\bigwedge_n e_{\mathcal{B}_n} = e_{\bigcap_n \mathcal{B}_n}.$$
- For a genenaral filtered collection $(\mathcal{B}_i)_i$ ordered by inclusion, we can only say that
$$\bigwedge_i e_{\mathcal{B}_i} = e_{\bigcap_i \mathcal{I}_{e_{\mathcal{B}_i}}} \sqsupseteq e_{\bigcap_i \mathcal{B}_i}.$$

Directed suprema and infima of idempotents

Proposition

Let $(e_i)_i$ be a filtered collection of dagger idempotents with dagger splittings $(\pi_i : X \to A_i, \iota_i : A_i \to X)$ and let $e : X \to X$ be another dagger idempotent with dagger splitting $(\pi : X \to A, \iota : A \to X)$ in a dagger category $\mathcal{C}$.

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Then $e = \bigvee_i e_i$ if and only if the following diagram is a limiting cone

$$
\begin{tikzcd}
 & & & & A \arrow[llll, "\pi_1 \circ \iota"'] \arrow[dl, "\pi_3 \circ \iota"] \\
A_1 & A_2 \arrow[l, "\pi_1 \circ \iota_2"] & A_3 \arrow[l, "\pi_2 \circ \iota_3"] & \dots \arrow[l] &
\end{tikzcd}
$$

L^2 preserves directed suprema and infima

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Let $(\mathcal{B}_n)_n$ be an increasing sequence of σ-subalgebras of A with join $\mathcal{B}$, i.e. a limit diagram in **Krn**

$$(X, \mathcal{B}_0, p) \xleftarrow[\pi_{0,1}]{} (X, \mathcal{B}_1, p) \xleftarrow[\pi_{1,2}]{} (X, \mathcal{B}_2, p) \longleftarrow \dots \longleftarrow (X, \mathcal{B}, p)$$

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From the previous two propositions, it follows that

$$L^2(X, \mathcal{B}_0, p) \xrightarrow{\pi_{0,1}^*} L^2(X, \mathcal{B}_1, p) \xrightarrow{\pi_{1,2}^*} L^2(X, \mathcal{B}_2, p) \longrightarrow \dots \longrightarrow L^2(X, \mathcal{B}, p)$$

is a colimit diagram.

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is a colimit diagram.
Since L^2 is a dagger functor, we also have that the following diagram is a limiting cone.

$$L^2(X, \mathcal{B}_0, p) \xleftarrow[(\pi^+_{0,1})^*]{} L^2(X, \mathcal{B}_1, p) \xleftarrow[(\pi^+_{1,2})^*]{} L^2(X, \mathcal{B}_2, p) \longleftarrow \dots \longleftarrow L^2(X, \mathcal{B}, p)$$

Martingale convergence I

Theorem

Let $(f_i)_i$ be an L^2-bounded martingale on a filtered probability space $(X, (\mathcal{B}_i)_i, \mathcal{B}, p)$, then there exists an almost surely unique $f \in L^2(X, \mathcal{B}, p)$ such that

$$\mathbb{E}[f|\mathcal{B}_i] = f_i \quad p\text{-almost surely.}$$

An L^2-bounded martingale forms the following cone in $\mathbf{Hilb}_{\leq 1}$.

$$\begin{tikzcd}
 & & & \mathbb{R} \arrow[dlll, "f_0"'] \arrow[dll, "f_1"'] \arrow[d, "f_2"] & & \\
L^2(X, \mathcal{B}_0, p) & L^2(X, \mathcal{B}_1, p) \arrow[l, "(\pi^+_{0,1})^*"] & & L^2(X, \mathcal{B}_2, p) \arrow[ll, "(\pi^+_{1,2})^*"] & \dots \arrow[l] & L^2(X, \mathcal{B}, p) \arrow[l]
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L^2(X, \mathcal{B}_0, p) & L^2(X, \mathcal{B}_1, p) \arrow[l, "(\pi^+_{0,1})^*"] & L^2(X, \mathcal{B}_2, p) \arrow[l, "(\pi^+_{1,2})^*"] & \dots \arrow[l] & L^2(X, \mathcal{B}, p) \arrow[l]
\end{tikzcd}$$

Remark: Dually we have a backwards martingale convergence result. Here we need to be careful and take $\mathcal{B}$ to be the infimum in the *idempotent order*.

Overview

	Dagger	Idempotents	Order	$\bigvee$	$\bigwedge$
Krn	Bayesian inverse	σ-subalgebras	$\lesssim$	$[\sigma(\underline{\bigcup_i \mathcal{B}_i})]$	$[\bigcap_n \mathcal{B}_n]$
$\mathbf{Hilb}_{\leq 1}$	adjoints	closed subspaces	$\subseteq$	$\overline{\bigcup_i A_i}$	$\bigcap_i A_i$
L^2	✓	-	✓	✓	✓

	Enrichment	Levi property
Krn		
$\mathbf{Hilb}_{\leq 1}$		
L^2		
$\mathbf{Ban}_{\leq 1}$		
L^n		

Topological enrichment

Let X and Y be topological spaces.

- Let $X \otimes Y$ be the set $X \times Y$ endowed with the final topology generated by the maps
$$((x,-) : Y \to X \times Y)_{x \in X}$$
and
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- Let $[X, Y]$ be the set of continuous maps toghether with the topology of poinwise convergence.

These form a monoidal closed structure on **Top**.

$\mathbf{Hilb}_{\leq 1}$ and $\mathbf{Ban}_{\leq 1}$ are enriched over $\mathbf{Top}$

For Banach spaces X and Y, we can give the set of 1-Lipschitz linear maps $\mathbf{Ban}_{\leq 1}(X, Y)$ a variation of the *strong operator topology*.

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For Banach spaces X and Y, we can give the set of 1-Lipschitz linear maps $\mathbf{Ban}_{\leq 1}(X, Y)$ a variation of the *strong operator topology*. A net $(f_\lambda : X \to Y)_\lambda$ converges to f if and only if for every $x \in X$,

$$\|f_\lambda(x) - f(x)\|_Y \to 0.$$

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This makes $\mathbf{Ban}_{\leq 1}$ a topologically enriched category. Similarly we can enrich $\mathbf{Hilb}_{\leq 1}$ over $\mathbf{Top}$.

Krn is enriched over **Top**

Let $(X, \mathcal{A}, p)$ and $(Y, \mathcal{B}, q)$ be essentially standard Borel probability spaces.

Krn is enriched over **Top**

Let $(X, \mathcal{A}, p)$ and $(Y, \mathcal{B}, q)$ be essentially standard Borel probability spaces. We say that a net $(k_\lambda : X \to Y)_\lambda$ of measure-preserving kernels converges to a measure-preserving kernel $k : X \to Y$ if and only if

$$\int_X |k_\lambda(B|x) - k(B|x)| p(\mathrm{d}x) \to 0 \quad \text{for all } B \in \mathcal{B}$$

This defines a topology on $\mathbf{Krn}((X, \mathcal{A}, p), (Y, \mathcal{B}, q))$ called the **one-sided topology** and enriches **Krn** over **Top**.

Krn is enriched over **Top**

Now we can look at the subspace of $\mathbf{Krn}((X, \mathcal{A}, p), (X, \mathcal{A}, p))$ of dagger idempotents.

Krn is enriched over **Top**

Now we can look at the subspace of $\mathbf{Krn}((X, \mathcal{A}, p), (X, \mathcal{A}, p))$ of dagger idempotents.

- The subspace is closed.
- A net $(e_{\mathcal{B}_\lambda} : X \to X)_\lambda$ converges to $e_{\mathcal{B}}$ if and only if for every $f \in L^1(X, \mathcal{A}, p)$:
$$\mathbb{E}[f \mid \mathcal{B}_\lambda] \to \mathbb{E}[f \mid \mathcal{B}] \quad \text{in } L^1.$$

L^n is enriched over **Top**

The following result implies that L^n is enriched, but is stronger and will be important in the next part.

Proposition

A net of measure-preserving kernels $(k_\lambda)_\lambda$ converges to k in $\mathbf{Krn}((X, \mathcal{A}, p), (Y, \mathcal{B}, q))$ *if and only if* $(k^*_\lambda)_\lambda$ converges to k^* in $\mathbf{Ban}_{\leq 1}(L^n(Y, \mathcal{B}, q), L^n(X, \mathcal{A}, p))$.

Overview

	Dagger	Idempotents	Order	$\bigvee$	$\bigwedge$
Krn	Bayesian inverse	σ-subalgebras	$\lesssim$	$[\sigma\left(\bigcup_i \mathcal{B}_i\right)]$	$[\bigcap_n \mathcal{B}_n]$
$\mathbf{Hilb}_{\leq 1}$	adjoints	closed subspaces	$\subseteq$	$\overline{\bigcup_i A_i}$	$\bigcap_i A_i$
L^2	✓	-	✓	✓	✓

	Enrichment	Levi property
Krn	one-sided	
$\mathbf{Hilb}_{\leq 1}$	strong operator	
L^2	✓	
$\mathbf{Ban}_{\leq 1}$	strong operator	
L^n	✓	

Idempotent Levi property

Recall that every increasing bounded sequence in $\mathbb{R}$ converges to its supremum.

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Definition

We say that a topologically enriched dagger category has the **idempotent Levi property** if every increasing (decreasing) net $(e_\lambda)_\lambda$ of dagger idempotents that has a supremum (infimum), converges topologically to its supremum (infimum)

Idempotent Levi property

Proposition

$\mathbf{Hilb}_{\leq 1}$ has the idempotent Levi property.

Proposition

Krn has the idempotent Levi property.

Proof: Let $(e_{\mathcal{B}_\lambda})_\lambda$ be an increasing net of dagger idempotents with a supremum.

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Krn has the idempotent Levi property.

Proof: Let $(e_{\mathcal{B}_\lambda})_\lambda$ be an increasing net of dagger idempotents with a supremum. Then $(e^*_{\mathcal{B}_\lambda})_\lambda$ is an increasing net with a supremum, hence it converges topologically.

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$\mathbf{Hilb}_{\leq 1}$ has the idempotent Levi property.

Proposition

Krn has the idempotent Levi property.

Proof: Let $(e_{\mathcal{B}_\lambda})_\lambda$ be an increasing net of dagger idempotents with a supremum. Then $(e^*_{\mathcal{B}_\lambda})_\lambda$ is an increasing net with a supremum, hence it converges topologically. Therefore, so does $(e_{\mathcal{B}_\lambda})_\lambda$.

Overview

	Dagger	Idempotents	Order	$\bigvee$	$\bigwedge$
Krn	Bayesian inverse	σ-subalgebras	$\lesssim$	$[\sigma\left(\underline{\bigcup_i \mathcal{B}_i}\right)]$	$[\bigcap_n \mathcal{B}_n]$
$\mathbf{Hilb}_{\leq 1}$	adjoints	closed subspaces	$\subseteq$	$\overline{\bigcup_i A_i}$	$\bigcap_i A_i$
L^2	✓	-	✓	✓	✓

	Enrichment	Levi property
Krn	one-sided	✓
$\mathbf{Hilb}_{\leq 1}$	strong operator	✓
L^2	✓	-
$\mathbf{Ban}_{\leq 1}$	strong operator	$\times$
L^n	✓	-

Martingale convergence II

Theorem

Let $(X, (\mathcal{B}_i)_i, \mathcal{B}, p)$ be a filtered probability space and let $f \in L^n(X, \mathcal{B}, p)$, then

$$\mathbb{E}[f \mid \mathcal{B}_\lambda] \to \mathbb{E}[f \mid \mathcal{B}] \quad \text{in } L^n.$$

Proof: Since $\bigvee_i e_{\mathcal{B}_i} = e_{\mathcal{B}}$, we have the idempotent Levi property that $e_{\mathcal{B}_i} \to e_{\mathcal{B}}$ and therefore $e^*_{\mathcal{B}_i} \to e^*_{\mathcal{B}}$. This implies the result.

Theorem

Let $(X, (\mathcal{B}_i)_i, \bigwedge_i \mathcal{B}_i, p)$ be a backwards filtered probability space and let $f \in L^n(X, \mathcal{B}, p)$, then

$$\mathbb{E}[f \mid \mathcal{B}_\lambda] \to \mathbb{E}[f \mid \mathcal{B}] \quad \text{in } L^n.$$

Conclusion

- Using enriched dagger categories we can prove the martingale convergence theorem.

Theorem

Let $n \geq 2$. Let $(f_i)_i$ be an L^n-bounded martingale on a filtered probability space $(X, (\mathcal{B}_i)_i, \mathcal{A}, p)$, then there exists an $f \in L^n(X, \mathcal{A}, p)$ such that:

1. $\mathbb{E}[f \mid \mathcal{B}_i] = f_i$, and
2. $f_i \to f$ in L^n.

- This generalizes to *vector-valued martingales* (see section 7 in [3]).

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